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N-soliton-type solutions of $SU(2)$ self-duality Yang–Mills equations in various spaces and their Bäcklund transformations

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Abstract. We show that Getmanov's N-soliton-type solutions in the self-duality Yang–Mills (SDYM) equations in the $(+---)$-signatured M^4-space can be derived in a unified way similar to that of deriving 't Hooft's instanton solutions in the $(++++)$-signatured E^4-space. Further, we show that such N-soliton-type solutions can also be constructed for the SDYM equations in the $(+-+-)$-signatured real D^4-space.

Recently, Getmanov [1] explicitly constructed N-soliton-type solutions to the (complex) *SU(2)* self-duality Yang–Mills (SDYM) equations (or, equivalently, the real *SU(2C)* SDYM equations) in the four-dimensional $(+---)$-signatured real M^4-space. Here, we shall show that such N-soliton-type solutions in M^4 can be derived in a unified way similar to that of deriving 't Hooft's instanton solutions in the $(++++)$-signatured complex E^4-space. We further show that such N-soliton-type solutions can also be constructed in the $(+-+-)$-signatured real D^4-space. The advantage of considering D^4-space is that both the SDYM equations and the D^4-space can be real; in contrast, the E^4-space has to be complexified in formulating the 't Hooft solutions. This important property of the D^4-space has been emphasized and made use of in a recent study [2] on the quantization of the SDYM equations.

First, we formulate the SDYM equations for various spaces in a unified way [3].

(i) For $x^\mu = (x^1, x^2, x^3, x^4)$ in E^4 with metric $(++++)$, let $y \equiv \frac{1}{\sqrt{2}}(x^1 + \mathrm{i}x^2)$, $\bar{y} \equiv \frac{1}{\sqrt{2}}(x^1 - \mathrm{i}x^2)$, $z \equiv \frac{1}{\sqrt{2}}(x^3 - \mathrm{i}x^4)$ and $\bar{z} \equiv \frac{1}{\sqrt{2}}(x^3 + \mathrm{i}x^4)$. Then, the SDYM equations $F_{12} = F_{34}$, $F_{13} = F_{42}$, $F_{14} = F_{23}$ become

$$F_{yz} = 0 \qquad F_{\bar{y}\bar{z}} = 0 \qquad F_{y\bar{y}} + F_{z\bar{z}} = 0. \tag{1}$$

(ii) For $x^\mu = (x^0, x^1, x^2, x^3)$ in M^4 with metric $(+---)$, let $y \equiv \frac{1}{\sqrt{2}}(x^1 + \mathrm{i}x^2)$, $\bar{y} \equiv \frac{1}{\sqrt{2}}(x^1 - \mathrm{i}x^2)$, $z \equiv \frac{1}{\sqrt{2}}(x^3 - x^0)$ and $\bar{z} \equiv \frac{1}{\sqrt{2}}(x^3 + x^0)$. Then, the complex SDYM equations $F_{01} = \mathrm{i}F_{23}$, $F_{02} = \mathrm{i}F_{31}$, $F_{03} = \mathrm{i}F_{12}$ again become equation (1).

(iii) For $x^\mu = (x^1, x^2, x^3, x^4)$ in D^4 with metric $(+-+-)$, let $y \equiv \frac{1}{\sqrt{2}}(x^1 + x^2)$, $\bar{y} \equiv \frac{1}{\sqrt{2}}(x^1 - x^2)$, $z \equiv \frac{1}{\sqrt{2}}(x^3 + x^4)$ and $\bar{z} \equiv \frac{1}{\sqrt{2}}(x^3 - x^4)$. The SDYM equations $F_{12} = -F_{34}$, $F_{13} = F_{42}$, $F_{14} = -F_{23}$ again become equation (1).

The first two equations in (1) can be immediately solved by $A_u = D^{-1}D_u$ ($u = y, z$) and $A_{\bar{u}} = \bar{D}^{-1}\bar{D}_{\bar{u}}$ ($\bar{u} = \bar{y}, \bar{z}$). If we define $J \equiv D\bar{D}^{-1}$ [4], the remaining equation in (1) becomes

$$\partial_{\bar{y}}(J^{-1}J_y) + \partial_{\bar{z}}(J^{-1}J_z) = 0 \tag{2}$$

which has been extensively studied during the period of the discoveries of the instanton and monopole solutions. Recall that a useful framework for discussing equation (2) is the parametrization

$$J = \frac{1}{\phi}\begin{bmatrix} 1 & \bar{\rho} \\ \rho & \phi^2 + \rho\bar{\rho} \end{bmatrix}$$

coupled with the so-called A_n-ansatz of Atiyah and Ward [5, 6] which is briefly recapitulated in the following.

It has been shown [6] that if a given J is a solution to equation (2), then so are the following two transforms of J:

$$J^I: \qquad \phi^I = \frac{\phi}{\phi^2 + \rho\bar{\rho}} \qquad \rho^I = \frac{\bar{\rho}}{\phi^2 + \rho\bar{\rho}} \qquad \bar{\rho}^I = \frac{\bar{\rho}}{\phi^2 + \rho\bar{\rho}} \tag{3}$$

$$J^B: \qquad \phi^B = \frac{1}{\phi} \qquad (\rho^B)_y = -\frac{1}{\phi^2}\bar{\rho}_{\bar{z}} \qquad (\rho^B)_z = \frac{1}{\phi^2}\bar{\rho}_{\bar{y}}$$

$$(\bar{\rho}^B)_{\bar{y}} = \frac{1}{\phi^2}\rho_z \qquad (\bar{\rho}^B)_{\bar{z}} = -\frac{1}{\phi^2}\rho_y. \tag{4}$$

The A_1-ansatz [7] is defined to be the set of J for which the following relations hold:

$$\rho_y = \phi_{\bar{z}} \qquad \rho_z = -\phi_{\bar{y}} \qquad \bar{\rho}_{\bar{y}} = \phi_z \qquad \bar{\rho}_{\bar{z}} = -\phi_y \tag{5}$$

$$(\text{and, hence, } \phi_{y\bar{y}} + \phi_{z\bar{z}} = 0). \tag{6}$$

The A_1-ansatz automatically satisfies equation (2). For $n \geqslant 2$, the A_n-ansatz can be defined inductively by I-transforming, followed by B-transforming, the A_{n-1}-ansatz:

$$A_1 \xrightarrow{I} \xrightarrow{B} A_2 \xrightarrow{I} \xrightarrow{B} A_3 \xrightarrow{I} \xrightarrow{B} \cdots.$$

We find it convenient to add an A_0-ansatz to the left end of the above sequence. A general representation of the A_0-ansatz is

$$J = \begin{bmatrix} \bar{h} - h + h\bar{h}\phi & \bar{h}\phi - 1 \\ h\phi + 1 & \phi \end{bmatrix} \tag{7}$$

where $h(\bar{h})$ is an arbitrary function of $y, z(\bar{y}, \bar{z})$, and ϕ satisfies equation (6).

A distinct characteristic for the A_0-ansatz solutions is that they are always associated with vanishing Lagrangian density. Indeed, every $F_{\mu\nu}$ can be shown to be proportional to a single nilpotent matrix, making $F_{\mu\nu}F_{\alpha\beta} = 0$ for any μ, ν, α, β, implying, in particular, $\operatorname{tr} F_{\mu\nu}F^{\mu\nu} = 0$. Hence, an A_0-ansatz solution by itself will usually not be one of the immediately interesting solutions we are seeking, but an appropriately chosen solution in the A_0-ansatz could be converted through the IB transformation into an interesting solution in the A_1-ansatz, such as 't Hooft's or Getmanov's solutions. For this reason, we will call a potentially useful A_0-ansatz solution a presolution. For example, we may take $\bar{h} = 1$, $h = -1$ and $\phi = 1 + f^{(N)}$ in equation (7), where

(i) for E^4-space,

$$f^{(N)} = \sum_{j=1}^{N} \frac{C_j}{(y - y_j)(\bar{y} - \bar{y}_j) + (z - z_j)(\bar{z} - \bar{z}_j)} \qquad (C_j > 0) \tag{8}$$

containing $5N$ parameters and satisfying $(\partial_1^2 + \partial_2^2 + \partial_3^2 + \partial_4^2) f^{(N)} = 0$;

(ii) for M^4-space,

$$f^{(N)} = \sum_{j=1}^{N} C_j \frac{\sinh w_j}{w_j} \mathrm{e}^{s_j} \qquad (C_j > 0) \tag{9}$$

where [1]

$$s_j = \varepsilon_j m_j [n_j \cdot (x - x_j)] \tag{10}$$

$$w_j = \{s_j^2 - m_j^2 (x - x_j)^2\}^{1/2} \tag{11}$$

with $\varepsilon_j = \pm 1$, $m_j > 0$, n_j = time-like unit vector in M^4, $f^{(N)}$ contains $9N$ continuous parameters and satisfies $(\partial_0^2 - \partial_1^2 - \partial_2^2 - \partial_3^2) f^{(N)} = 0$; and

(iii) for D^4-space,

$$f^{(N)} = \sum_{j=1}^{N} C_j I_0(s_j) I_0(w_j) \qquad (C_j > 0) \tag{12}$$

where I_0 is the standard modified Bessel function I_0

$$s_j = m_j \left\{ \sum_{a=1}^{2} [n_j^a \cdot (x - x_j)]^2 \right\}^{1/2} \qquad m_j > 0 \tag{13}$$

$$w_j = m_j \left\{ \sum_{a=1}^{2} [k_j^a \cdot (x - x_j)]^2 \right\}^{1/2} \tag{14}$$

with $(n_j^1, k_j^1, n_j^2, k_j^2)$ forming an orthonormal basis in D^4, $(n_j^a)^2 = 1$ and $(k_j^a)^2 = -1$. $f^{(N)}$ contains $12N$ parameters and satisfies $(\partial_1^2 - \partial_2^2 + \partial_3^2 - \partial_4^2) f^{(N)} = 0$. This solution is derived using an analogy with Getmanov's solution (9)–(11).

The A_0-ansatz solution

$$J^{(N)} = \begin{bmatrix} 1 - f^{(N)} & f^{(N)} \\ -f^{(N)} & 1 + f^{(N)} \end{bmatrix} \tag{15}$$

in each of the above three cases is a presolution, since it can be readily IB transformed into the N-instanton or N-soliton-type solutions in the A_1-ansatz.

A common criterion for choosing the solutions to equations (8), (9), and (12) is that, for these solutions in the A_1-ansatz (with $\phi = 1 + f^{(N)}$), there exists a gauge in which every component of the gauge field A_μ^a is manifestly regular everywhere in the real space. This assertion can be doubly checked by inspecting the analyticity of $\mathrm{tr}\, F_{\mu\nu}F^{\mu\nu}$, which can be shown to be proportional to $\Box\Box \log\phi$.

Now we briefly describe the 'one-soliton'-type solution in D^4-space. (Note that we call these solutions soliton-type solutions because of a lack of better names. These solutons do not necessarily have all the nice properties of the usual solitons.) For simplicity, we may choose $n_1^1 = (1,0,0,0)$, $n_1^2 = (0,0,1,0)$, $k_1^1 = (0,1,0,0)$, $k_1^2 = (0,0,0,1)$ and $x_1 = (0,0,0,0)$ in equations (13) and (14). Then

$$s_1 = m_1[(x^1)^2 + (x^3)^2]^{1/2} \qquad w_1 = m_1[(x^2)^2 + (x^4)^2]^{1/2}$$

and

$$f^{(1)} = C_1 I_0(s_1) I_0(w_1).$$

The *SU(2)* one-soliton-type solution in D^4-space is then explicitly given by

$$A_\mu^a = \eta_{a\mu\nu}\partial^\nu \log[1 + f^{(1)}] \qquad (a = 1,2,3;\ \mu = 1,2,3,4)$$

where

$$\eta_{a\mu\nu} = (g_{a\mu}g_{\nu 4} - g_{a\nu}g_{\mu 4} - \varepsilon_{a\mu\nu 4}) \cdot \zeta_a$$

with

$$\zeta_a = 1 \quad \text{for } a = 2 \qquad \zeta_a = -i \quad \text{for } a = 1 \text{ or } 3 \qquad \varepsilon_{1234} = +1.$$

Here, $\eta_{a\mu\nu}$ is just the D^4-space version of the 't Hooft tensor. Thus, A_μ^a is manifestly regular everywhere in D^4-space. Note that if we replace the gauge group *SU(2)* by *SU(1,1)*, A_μ^a would become completely real.

An interesting question is whether there are Bäcklund transformations (BTs) for the solutions of the SDYM equations, i.e. whether one can systematically generate new solutions to the SDYM equations starting from a given solution? We have just mentioned two such examples in the I- and B-transformations described in equations (3) and (4). There have been several other successful constructions of such BTs by Belavin *et al* [8], Forgacs *et al* [9], Ueno *et al* [10], Mason *et al* [11] and by the present authors [12], all of which, although different in formulation, follow more or less the same spirit in their schemes and are probably related to each other in some implicit way.

In particular, we would like to know whether BT can actually bring the N-soliton solution into the $(N+1)$-soliton solution. This turns out to be a non-trivial problem. However, if we do not attack the problem directly in terms of the N-soliton solutions *per se*, but instead deal with their presolutions in the A_0-ansatz, then the problem becomes much more manageable. Forgacs *et al* [9] and Ueno *et al* [10] have shown how this can be achieved for 't Hooft's N-instanton solution in E^4 and in the following we will show the same for Getmanov's N-soliton-type solution in M^4. However, we have difficulties in carrying out a supposedly corresponding procedure for the N-soliton-type solution in D^4.

First, in E^4-space, we organize the procedure of generating $(N+1)$-soliton solutions from N-soliton solutions. Instead of the presolution $J^{(N)}$ of equation (15), we will replace it with an even simpler version from now on which is just a similarity transform of equation (15) (no longer in the A_0-ansatz):

$$J^{(N)} \to \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} J^{(N)} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & f^{(N)} \\ 0 & 1 \end{bmatrix}.$$

Thus, we want to describe a scheme that can bring

$$J^{(N)} \equiv \begin{bmatrix} 1 & f^{(N)} \\ 0 & 1 \end{bmatrix} \tag{16}$$

into $J^{(N+1)}$. The whole BT can be schematically represented as

$$\begin{array}{ccc}
N\text{-soliton solution} & \longrightarrow & (N+1)\text{-soliton solution} \\
\Big\updownarrow IB & & \Big\updownarrow IB \\
\text{presolution} & & \text{presolution} \\
\Big\updownarrow \textit{gauge tr.} & & \Big\updownarrow \textit{gauge tr.} \\
\begin{array}{l}\text{similarity-transformed} \\ \text{presolutions}\end{array} & \xrightarrow[\text{transform}]{\text{RH}} & \begin{array}{l}\text{similarity-transformed} \\ \text{presolutions.}\end{array}
\end{array}$$

Given the solution $J = J^{(N)}$, equation (16), we first solve for its wavefunction $Y(\lambda)$ which is a 2×2 invertible matrix depending on a parameter λ and satisfying the linear equations

$$\begin{aligned} (D_1 + J^{-1}J_y)Y(\lambda) &= 0 \qquad D_1 \equiv \partial_y - \lambda\partial_{\bar{z}} \\ (D_2 + J^{-1}J_z)Y(\lambda) &= 0 \qquad D_2 \equiv \partial_z + \lambda\partial_{\bar{y}}. \end{aligned} \tag{17}$$

Note that $Y(\lambda)$ is determined only up to a multiplication from right by an arbitrary invertible matrix $g(\lambda)$ satisfying

$$D_i g(\lambda) = 0 \qquad i = 1, 2. \tag{18}$$

Also note that $Y(0)^{-1}$ may be taken to be a solution for J. Ueno and Nakamura's BT [10] is in fact a Riemann–Hilbert (RH) transformation for the wavefunction $Y(\lambda)$, which we now briefly describe.

Let C be a closed contour in the complex λ-plane and $U(\lambda)$ a chosen non-singular matrix function analytic on C, satisfying $D_i U(\lambda) = 0$. Let $H(\lambda) \equiv Y(\lambda)U(\lambda)[Y(\lambda)]^{-1}$ for λ on C, where $Y(\lambda)$ is the input wavefunction provided by (17). The RH problem in this case is to find a decomposition $H(\lambda) = [X_+(\lambda)]^{-1} \cdot X_-(\lambda)$ such that $X_+(\lambda)$ is holomorphic and non-singular inside of C; and $X_-(\lambda)$ outside of C. If the RH problem is solved, then define the output wavefunction $\tilde{Y}'(\lambda)$ by

$$\tilde{Y}'(\lambda) \equiv X_+(\lambda)Y(\lambda) \qquad \text{for } \lambda \text{ inside of } C.$$

Then $J' = \tilde{Y}'(0)^{-1}$ can be shown to be a new solution to the SDYM equations, and, in the meantime, $\tilde{Y}'(\lambda)$ is its corresponding wavefunction.

For the N-instanton problem in E^4, Ueno and Nakamura [10] have chosen (we have changed some of their notation)

$$U(\lambda) = 1 + \frac{a'_{N+1}}{\lambda - \alpha_{N+1}} P \tag{19}$$

and

$$Y(\lambda) = \begin{bmatrix} 1 + \phi^{(N+1)}(\lambda) & \sum_{j=1}^{N} \frac{a'_j}{\lambda - \alpha_j}[\phi^{(j)}(\alpha_j) - \phi^{(N+1)}(\lambda)] \\ 0 & 1 \end{bmatrix} \tag{20}$$

where

$$P = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad a'_j = \frac{C_j}{z - z_j} \qquad \alpha_j = \frac{\bar{y} - \bar{y}_j}{z - z_j}$$

and

$$\phi^{(j)}(\lambda) = \frac{\lambda}{\lambda(y - y_j) + (\bar{z} - \bar{z}_j)} \qquad j = 1, 2, 3, \ldots.$$

It is easy to check that $\sum_{j=1}^{N}(a'_j/\alpha_j)\phi^{(j)}(\alpha_j) = f^{(N)}$ of equation (8) and, hence, $Y(0)^{-1} = J^{(N)}$. The result of RH-decomposing $H(\lambda)$ is such that the following output wavefunction is obtained which corresponds to a new solution of the SDE $J' = \tilde{Y}'(0)^{-1} = J^{(N+1)}$:

$$\tilde{Y}'(\lambda) = \begin{bmatrix} 1 + \phi^{(N+1)}(\lambda) & \sum_{j=1}^{N+1} \frac{a'_j}{\lambda - \alpha_j}[\phi^{(j)}(\alpha_j) - \phi^{(N+1)}(\lambda)] \\ 0 & 1 \end{bmatrix}. \tag{21}$$

The above procedure can, of course, be iterated, but, before doing so, the new wavefunction $\tilde{Y}'(\lambda)$ must be modified by replacing $\phi^{(N+1)}(\lambda)$ in (21) with $\phi^{(N+2)}(\lambda)$, in conformity with the general form equation (20) for the next-step input wavefunction. This modification of the wavefunction is simply an exercise in the free choice of wavefunctions discussed between equations (17) and (18).

Now, in M^4-space, we describe a similar procedure for transforming the N-soliton-type solution into the $(N+1)$-soliton-type solution. We first introduce a set of notation for convenience. From equations (10) and (11), we define $p_j, q_j, r_j, \bar{r}_j$ by

$$\begin{aligned} s_j &= \varepsilon_j m_j [n_j \cdot (x - x_j)] \\ &\equiv \varepsilon_j m_j [p_j(\bar{z} - \bar{z}_j) - q_j(z - z_j) - r_j(\bar{y} - \bar{y}_j) - \bar{r}_j(y - y_j)] \end{aligned} \tag{22}$$

where

$$p_j q_j - r_j \bar{r}_j = \tfrac{1}{2} \qquad \text{since } n_j^2 = 1.$$

Then

$$w_j^2 = s_j^2 + m_j^2[2(y - y_j)(\bar{y} - \bar{y}_j) + 2(z - z_j)(\bar{z} - \bar{z}_j)]. \tag{23}$$

Suppose a solution $J = J^{(N)}$, equation (16), is given where $f^{(N)}$ is defined by equations (9), (22), and (23). Similarly to equations (19) and (20), we may take

$$U(\lambda) = 1 + a'_{N+1}\left[\frac{1}{\lambda - \alpha_{N+1}} - \frac{1}{\lambda - \beta_{N+1}}\right] P \tag{24}$$

and

$$Y(\lambda) = \begin{bmatrix} 1+\phi^{(N+1)}(\lambda) & \sum_{j=1}^{N} a'_j \left[\frac{\phi^{(j)}(\alpha_j)-\phi^{(N+1)}(\lambda)}{\lambda-\alpha_j} - \frac{\phi^{(j)}(\beta_j)-\phi^{(N+1)}(\lambda)}{\lambda-\beta_j}\right] \\ 0 & 1 \end{bmatrix} \tag{25}$$

where

$$P = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad \alpha_j = \frac{1}{2D_j}(A_j - B_j) \qquad \beta_j = \frac{-1}{2D_j}(A_j + B_j)$$

with

$$D_j \equiv p_j(y-y_j) + r_j(z-z_j) \qquad A_j \equiv w_j/m_j$$

$$B_j \equiv p_j(\bar z - \bar z_j) + q_j(z-z_j) - r_j(\bar y - \bar y_j) + \bar r_j(y-y_j)$$

and

$$a'_j = \frac{C_j}{2m_j D_j(\alpha_j - \beta_j)} = \frac{C_j}{2w_j}$$

$$\phi^{(j)} = \lambda\varepsilon_j \exp(2\varepsilon_j m_j\{p_j[\lambda(y-y_j)+(\bar z-\bar z_j)] + r_j[\lambda(z-z_j)-(\bar y-\bar y_j)]\}).$$

Note that

$$a'_j\left[\frac{\phi^{(j)}(\alpha_j)}{\alpha_j} - \frac{\phi^{(j)}(\beta_j)}{\beta_j}\right] = C_j\frac{\sinh w_j}{w_j}e^{s_j}.$$

Carrying out Ueno and Nakamura's procedure eventually leads to the output wavefunction

$$\tilde Y'(\lambda) = \begin{bmatrix} 1+\phi^{(N+1)}(\lambda) & \sum_{j=1}^{N+1} a'_j \left[\frac{\phi^{(j)}(\alpha_j)-\phi^{(N+1)}(\lambda)}{\lambda-\alpha_j} - \frac{\phi^{(j)}(\beta_j)-\phi^{(N+1)}(\lambda)}{\lambda-\beta_j}\right] \\ 0 & 1 \end{bmatrix}$$

corresponding to a new solution $J' = \tilde Y'(0)^{-1} = J^{(N+1)}$, which is desired. We have found the above particular choice of α_j, β_j and $\phi^{(j)}(\lambda)$ through a generalization of the results in [12] on a BT for monopole solutions. In fact, α_j and β_j are the two roots of the following algebraic equation in λ:

$$\left(p_j + \frac{1}{\lambda}\bar r_j\right)[\lambda(y-y_j)+(\bar z-\bar z_j)] + \left(r_j + \frac{1}{\lambda}q_j\right)[\lambda(z-z_j)-(\bar y-\bar y_j)] = 0.$$

For the D^4-space, a similar procedure to the above for transforming the N-soliton-type solution into the $(N+1)$-soliton-type solution should, in principle, be achieved straightforwardly. However, so far we have not been able to find the correct wavefunction $\phi^{(j)}(\lambda)$ and poles $\alpha_j(y,z,\bar y,\bar z)$ and $\beta_j(y,z,\bar y,\bar z)$ for this to succeed.

References

[1] Getmanov B S 1990 *Phys. Lett.* **244B** 455
[2] Chau L-L and Yamanaka I 1992 *Phys. Rev. Lett.* **68** 1807
[3] Yang C N 1977 *Phys. Rev. Lett.* **38** 1377
[4] Brihaye Y, Fairlie D B, Nuyts J and Yates R G 1978 *J. Math. Phys.* **19** 2528
[5] Atiyah M F and Ward R S 1977 *Commun. Math. Phys.* **55** 117
[6] Corrigan E F, Fairlie D B, Goddard P and Yates R G 1978 *Commun. Math. Phys.* **58** 223
[7] Corrigan E F and Fairlie D B 1977 *Phys. Lett.* **67B** 69
't Hooft G unpublished
Wilczek F 1977 *Quark Confinement and Field Theory* ed D Stamp and D Weingartan (New York: Wiley)
[8] Belavin A A and Zakharov V E 1978 *Phys. Lett.* **73B** 53
[9] Forgacs P, Horvath Z and Palla L 1981 *Phys. Rev.* D **23** 1876; 1982 *Phys. Lett.* **109B** 200; 1983 *Nucl. Phys.* B **229** 77
[10] Ueno K and Nakamura Y 1982 *Phys. Lett.* **109B** 273
[11] Mason L, Chakravarty S and Newman E T 1988 *J. Math. Phys.* **29** 1005
[12] Chau L-L, Shaw J C and Yen H C 1989 *Int. J. Mod. Phys.* A **4** 2715